Exponential ergodicity of generalized branching processes

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The 18th Workshop on Markov Process and Related Topics

August 1, 2023

Outline

- Classical continuous state branching processes
- Some known generalized processes
- Our result

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Continuous state branching processes

A continuous state branching process (with immigration) (CBI in short) can be constructed by the positive strong solution of (Dawson–Li,'06/'12):

$$\begin{aligned} Y(t) &= Y(0) - b\int_0^t Y(s)\,\mathrm{d}s + \sqrt{2c}\int_0^t\int_0^{Y(s)} W(\mathrm{d}s,\mathrm{d}u) \\ &\quad + \int_0^t\int_0^\infty\int_0^{Y(s-)} z\,\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \eta(t), \end{aligned} \tag{1}$$

- $W(\mathrm{d}s,\mathrm{d}u)$ is a Gaussian white noise.
- $\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ is a compensated Poisson random measure.
- $\eta(t)$ is a positive increasing Lévy process with $-\log\mathbb{E}(\mathrm{e}^{-\lambda\eta(1)}) = \psi(\lambda)$,

$$\psi(\lambda) = \beta\lambda + \int_0^\infty (1-\mathrm{e}^{-\lambda u})\,\nu(\mathrm{d}u).$$

Continuous state branching processes: Ergodicity

- For $x \geq 0$, let $\{Y^x(t) : t \geq 0\}$ denote the solution of (1) with initial value $Y^x(0) = x$. If $y \geq x \geq 0$, then $\{Y^y(t) - Y^x(t) : t \geq 0\}$ is a CB process.

- For $t > 0$ and $y \geq x \geq 0$ $(P_t(x, \cdot) = \delta_x P_t)$

$$\|P_t(y, \cdot) - P_t(x, \cdot)\|_{\mathrm{var}} \leq \mathbb{P}(Y^y(t) - Y^x(t) \neq 0). \tag{2}$$

From (2): strong Feller property, exponential ergodicity; see Li–Ma('15).

- Affine framework.

Coupling methods: Friesen–Jin–Rüdiger('20), Bao–Wang('23),C.–Li('22).

Meyn–Tweedie's method: Barczy et al. ('14), Zhang–Glynn('18).

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Some known generalized processes I

- CBI processes in Lévy random environments $L(t)$.

$$
\begin{aligned}
Y(t) &= Y(0) - b\int_0^t Y(s)\,\mathrm{d}s + \sqrt{2c}\int_0^t\int_0^{Y(s)} W(\mathrm{d}s,\mathrm{d}u) \\
&\quad + \int_0^t\int_0^\infty\int_0^{Y(s-)} z\,\tilde{M}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) + \eta(t) + \int_0^t Y(s-)\mathrm{d}L(t);
\end{aligned}
$$

see Palau–Pardo('17), He–Li–Xu('18).

- Coupling methods: Friesen et al. ('23), C.–Fang–Zheng ('22).

Some known generalized processes II

- Nonlinear branching processes

$$Y(t) = Y(0) + \int_0^t \gamma_0(Y(s))\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(Y(s))}\,\mathrm{d}B(s)$$
$$+ \int_0^t \int_0^{\gamma_2(Y(s-))} \int_0^\infty z\,\tilde{M}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}z);$$

see Li ('18), Li–Yang–Zhou ('19).

∗ $\gamma_0(x) = a - bx, \gamma_i(x) = c_i x, i = 1, 2,$ CBI process.

∗ $\gamma_0(x) = \theta x - \gamma x^2, \gamma_i(x) = c_i x, i = 1, 2,$ Logistic branching process/ CB process with competition. Lambert('05), Pardoux('16).

Nonlinear CB process: Ergodicity

• Friesen et al. ('23): L^1-Wasserstein distance under uniformly dissipative condition on γ_0:

$$\gamma_0(x) - \gamma_0(y) \leq -k(x-y), \quad 0 \leq y \leq x.$$

• Li–Wang ('20): L^1-Wasserstein distance and total variation distance under dissipative condition for large distance on γ_0:

$$\gamma_0(x) - \gamma_0(y) \leq \begin{cases} \Phi(x-y), & 0 \leq x-y \leq l_0, \\ -k(x-y), & x-y > l_0. \end{cases}$$

Nonlinear CB process: Ergodicity

- Li et al. ('23+): $\gamma_0(x) = -bx - g(x) + \beta$, $\gamma_1(x) = \gamma_2(x) = x$.

Lyapunov condition: There exists a function $V \geq 1$ with $V(x) \to \infty$ as $x \to \infty$ and $(C_0, C_1 > 0)$

$$LV(x) \leq C_0 - C_1 V(x), \quad x \geq 0.$$

- A key condition for exponential ergodicity in total variation distance of Markov process; see Meyn–Tweedie ('93), Wang ('08).
- V-weighted total variation distance: e.g.

$$\Phi(x,y) := |x-y|(V(x)+V(y)), \Phi(x,y) := (V(x)+V(y)+1)\mathbf{1}_{\{x \neq y\}};$$

Hairer–Mattingly ('11), Eberle–Guillin-Zimmer ('19), Huang–Majka–Wang ('22).

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Our result–Motivation

- Cattiaux–Méléard ('10): studied the existence and uniqueness of QSD of stochastic Lotka–Volterra systems:

$$X(t) = X(0) + \int_0^t \gamma_0(X(s), Y(s))\,\mathrm{d}s + \int_0^t \sqrt{\vartheta_1 X(s)}\,\mathrm{d}B_1(s),$$
$$Y(t) = Y(0) + \int_0^t \gamma_1(X(s), Y(s))\,\mathrm{d}s + \int_0^t \sqrt{\vartheta_2 Y(s)}\,\mathrm{d}B_2(s),$$

where $\gamma_0(x, y) = r_1 x - c_{11}x^2 - c_{12}xy$ and $\gamma_1(x, y) = r_2 y - c_{22}y^2 - c_{21}xy$ with $r_1, r_2, c_{11}, c_{22}, \vartheta_1, \vartheta_2 > 0$.

Our result–Model

Consider Lotka-Volterra systems driven by pure-jump processes,

$$X(t) = X(0) + \int_0^t \gamma_0(X(s), Y(s))\, \mathrm{d}s + \int_0^t \int_0^\infty \int_0^{X(s-)} z\, \tilde{M}_1(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

$$Y(t) = Y(0) + \int_0^t \gamma_1(X(s), Y(s))\, \mathrm{d}s + \int_0^t \int_0^\infty \int_0^{Y(s-)} z\, \tilde{M}_2(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

with finite measures $(z \wedge z^2)\, m_i(\mathrm{d}z)$, $i = 1, 2$ and

$$\gamma_0(x, y) = r_1 x - c_{11} x^2 - c_{12} xy + p, \quad \gamma_1(x, y) = r_2 y - c_{22} y^2 - c_{21} xy + q$$

satisfying $c_{11}, c_{22}, p, q > 0$.

Our result–Model

The generator L is given by

$$\begin{aligned} Lf(x,y) &= \gamma_0(x,y)f'_x(x,y) + x\int_0^\infty D_{(z,0)}f(x,y)\, m_1(\mathrm{d}z) \\ &\quad + \gamma_1(x,y)f'_y(x,y) + y\int_0^\infty D_{(0,z)}f(x,y)\, m_2(\mathrm{d}z), \end{aligned}$$

where $D_{(a,b)}f(x,y) := f(x+a,y+b) - f(x,y) - af'_x(x,y) - bf'_y(x,y)$.

Our result–Assumptions

(a1) There exists $V(z) \geq 1$ with $V(z) \to \infty$ as $x + y \to \infty$ and $\lambda_1, \lambda_2 > 0$ such that

$$LV(z) \leq \lambda_1 - \lambda_2 V(z).$$

where $z = (x, y)$.

(a2) There exists $k_0 > 0$ such that $(|z - \tilde{z}| := |x - \tilde{x}| + |y - \tilde{y}|)$

$$\frac{(\gamma_0(x,y) - \gamma_0(\tilde{x},\tilde{y}))(x - \tilde{x})}{|x - \tilde{x}|} + \frac{(\gamma_1(x,y) - \gamma_1(\tilde{x},\tilde{y}))(y - \tilde{y})}{|y - \tilde{y}|}$$
$$\leq |r_1||x - \tilde{x}| + |r_2||y - \tilde{y}| + k_0|z - \tilde{z}|(V(z) + V(\tilde{z})).$$

Our result–Assumptions

(a3) There exists $\lambda_* > 0$ such that $\phi_1(\lambda_*) \wedge \phi_2(\lambda_*) > 0$, where

$$\phi_i(\lambda) = -r_i\lambda + \int_0^\infty (\mathrm{e}^{-\lambda z} - 1 + \lambda z)\, m_i(\mathrm{d}z).$$

(a4) There exists $c_0 > 0$ and $\kappa_0 > 0$ such that

$$m_{1,x}(\mathbb{R}_+) \wedge m_{2,x}(\mathbb{R}_+) \geq \kappa_0, \quad |x| \leq c_0,$$

where $\mu_x(\cdot) := \mu \wedge (\delta_x * \mu)$.

Our result

Theorem

Assume that (a1)–(a4) hold. Then there exists $K = K(\lambda_1, \lambda_2, \phi_1(\lambda_*), \phi_2(\lambda_*), \kappa_0, c_0) > 0$, such that for $k_0 \in (0, K]$, $\{(X(t), Y(t)) : t \geq 0\}$ is exponential ergodic in the distance

$$W_V(\rho_1, \rho_2) = \inf_{\pi \in \mathcal{C}(\rho_1, \rho_2)} \int_{\mathbb{R}_+^4} [V(z) + V(\tilde{z}) + 1] \mathbf{1}_{\{z \neq \tilde{z}\}} \pi(\mathrm{d}z, \mathrm{d}\tilde{z}).$$

Our result–Sketch of the proof

- Construct the coupling process $(X, Y, \tilde{X}, \tilde{Y})$ and generator $\tilde{L}$.
- Define $l_0 := \sup_{(z,\tilde{z})\in S_0} (|z-\tilde{z}|) + 1$ and

$$S_0 = \Big\{(z,\tilde{z}) : \lambda_2(V(z)+V(\tilde{z})) \leq 6\lambda_1\Big\}.$$

- Choice proper $\varepsilon > 0$ and function ψ,

$$f(z,\tilde{z}) = \Big[V(z)+V(\tilde{z})+\varepsilon\psi(z,\tilde{z},l_0)\Big]\mathbf{1}_{\{z\neq\tilde{z}\}}.$$

- For $|z-\tilde{z}| \in (0, l_0]$, we have

$$\begin{aligned}\tilde{L}f(z,\tilde{z}) &\leq 2\lambda_2 - \lambda_1(V(z)+V(\tilde{z})) + \varepsilon\Big[\tilde{C}_0 k_0(V(z)+V(\tilde{z})) - \tilde{C}_1\Big] \\ &\leq -\tilde{C}_3(V(z)+V(\tilde{z})+1).\end{aligned}$$

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To appear in *Ann. Inst. Henri Poincaré Probab. Stat.* 2023+.

Thank you!